Group Theory in Music

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Introduction

While group theory is traditionally studied in the world of abstract algebra and Pure Mathematics, it has various applications in the real world, including Physics, Chemistry and Cryptography. Here I will highlight some of the connections group theory has with Music. Links between Maths and Music can be found as far back in history as Pythagoras. I will focus on a more recent theory known as Neo-Riemannian theory, which was introduced in the 19th century by Hugo Riemann (1). Neo-Riemannian theory provides a link between some well known harmonic concepts in music, and group theory.

Group Theory

Definition 1: A *group* is a set G equipped with a binary operation $*$ such that the following hold:

1. **Closure**: For all $x, y \in G$, $x * y \in G$
2. **Associativity**: For all $x, y, z \in G$, $(x * y) * z = x * (y * z)$
3. **Identity**: There exists a unique $e \in G$, called the *identity element*, such that for all $x \in G$, $x * e = e * x = x$
4. **Inverses**: For all $x \in G$, there exists $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$

One of the simplest examples is the set of integers $\mathbb{Z}$ under addition. Here is another example which we all use in our day-to-day lives:

Example 1: The group of integers under addition modulo n are a group. For example $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$ can be represented by reading the time for A.M. and P.M. on a clock!

Chromatic Scale

In Western music, a *chromatic scale* is an ascending scale of all 12 notes, increasing each time by a semitone. We can label these notes as follows:

Note	C	C♯/D♭	D	D♯/E♭	E	F	F♯/G♭	G	G♯/A♭	A	A♯/B♭	B
Label	0	1	2	3	4	5	6	7	8	9	10	11

You may notice this scale is equivalent to the group $\mathbb{Z}_{12}$, where our binary operation is equivalent to adding or removing sharps or flats.

Major and Minor Chords

Let's consider the combinations of playing 3 notes in the chromatic scale simultaneously. These combinations are known in music as *triads*. There are $\binom{12}{3} = 220$ options for these, though we will focus on the more traditional choices known as major and minor chords. A *major* chord consists of a root, major 3rd and perfect 5th, whereas a *minor* chord consists of a root, minor 3rd and perfect 5th. If we denote (x, y, z) to be the three notes of a chord, we can represent all major and minor chords by the following set:

$$\mathcal{M} = \{(x, x+3, x+7), (y, y+4, y+7) \mid x, y \in \mathbb{Z}_{12}\}$$

using our notation from the chromatic scale. Note here we are looking at an unordered set, for example:

$$(0, 4, 7) = (4, 7, 0) = (7, 0, 4)$$

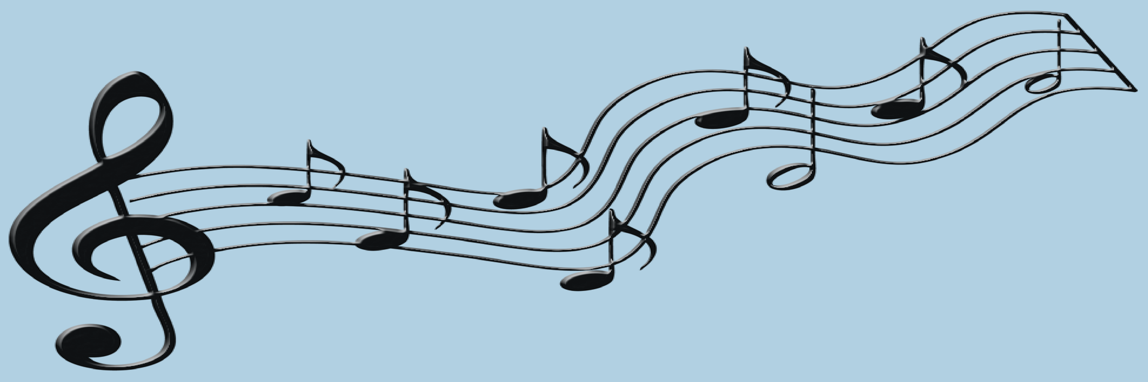
all represent a C Major Chord.

References

[1] F. Aceff-Sánchez, O. A. Agustín-Aquino, J. D. Plessis, E. Lluis-Puebla, J. Du Plessis, and M. Montiel, "An Introduction to Group Theory with applications to Mathematical Music Theory," *Serie: Textos*, vol. 15, 2012.

[2] A. S. Crans, T. M. Fiore, and R. Satyendra, "Musical actions of dihedral groups," *American Mathematical Monthly*, vol. 116, no. 6, pp. 479–495, 2009.

[3] E. B. Roon, "That strikes a chord! An illustration of permutation groups in music theory," pp. 1–16.



The $\mathcal{PLR}$ Group

From now on, we will denote $(a, b, c) \in \mathcal{M}$ as a minor chord, and $(A, B, C) \in \mathcal{M}$ as a major chord. We define the *parity* of a chord to be whether a chord in $\mathcal{M}$ is either major or minor. We can define three functions which act on the set $\mathcal{M}$, which are analogous to common terminology in music. When combined, these three functions form a group!

Two chords are *parallel* if they have the same letter name, but opposite parity. In particular, we can define the following function to describe how to find corresponding parallel chords:

$$\begin{aligned} \mathcal{P} &: \mathcal{M} \to \mathcal{M} \\ (a, b, c) &\mapsto (a, b+1, c) \\ (A, B, C) &\mapsto (A, B-1, C) \end{aligned}$$

For example, $\mathcal{P}(C \text{ minor}) = \mathcal{P}((0, 3, 7)) = (0, 4, 7) = C$ major.

The *leading note* is the note obtained when we move one semitone below the original note. This definition leads to another transformation known as the *leading tone exchange*, which preserves the third in the triad, and moves the remaining note by a semitone. In particular, we define this as:

$$\begin{aligned} \mathcal{L} &: M \to M \\ (a, b, c) &\mapsto (c+1, a, b) \\ (A, B, C) &\mapsto (B, C, A-1) \end{aligned}$$

Two chords are *relative* if they are of opposite parity and share the same key signature, i.e. the same assignment of sharps/flats. Any minor chord is three semitones below its relative major chord, and so we can define the following to represent moving from a chord to its relative major/minor:

$$\begin{aligned} \mathcal{R} &: M \to M \\ (a, b, c) &\mapsto (b, c, a-2) \\ (A, B, C) &\mapsto (C+2, A, B) \end{aligned}$$

These can be viewed pictorially below by a *Tonnetx diagram* (1). The numbers represent the pitch of the notes based on the chromatic scale, and each triangle represents either a major or minor chord. The functions $\mathcal{P}, \mathcal{L}$ and $\mathcal{R}$ allow us to flip between triangles. The word 'Tonnetx' means 'tone network', and this diagram also represents other key properties from music. For example, each horizontal axis shows the *circle of fifths* (which also can be represented by $\mathbb{Z}_{12}$!).

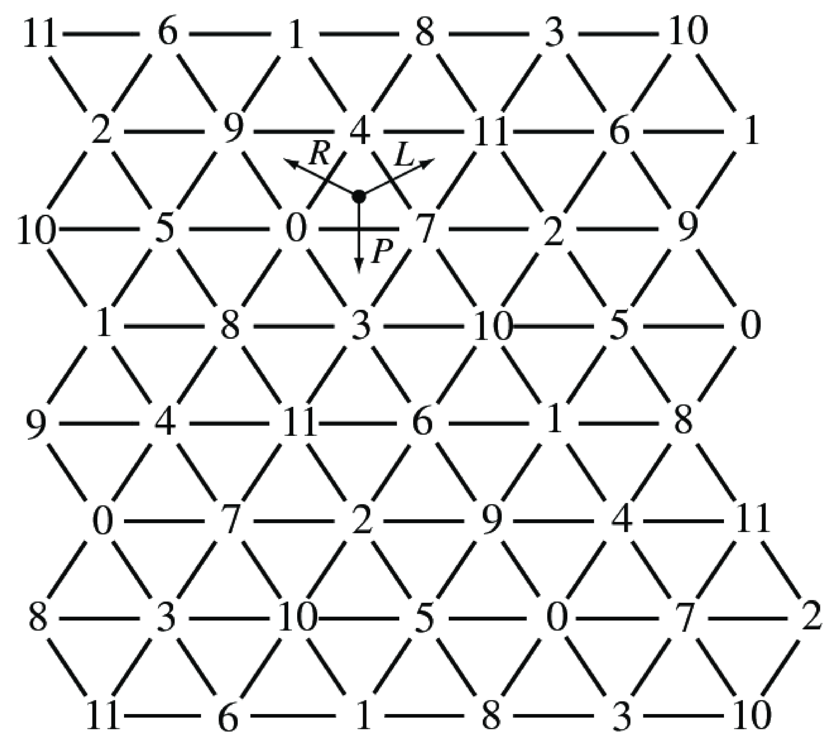


Figure 1: Tonnetx diagram

Theorem 1: The set $G = \langle \mathcal{P}, \mathcal{L}, \mathcal{R} \rangle$ generated by the $\mathcal{P}, \mathcal{L}$ and $\mathcal{R}$ functions form a group, with binary operation defined as composition of functions.

Proof of Theorem 1: We simply need to check the four group axioms defined in Definition 1.

Closure: One would need to check that any composition of the $\mathcal{P}, \mathcal{L}$ and $\mathcal{R}$ functions gives another $\mathcal{P}, \mathcal{L}$ or $\mathcal{R}$ function (this is a rather tedious check which I leave for any interested reader!)

Associative: Composition of functions is always associative.

Identity: We can define an identity function $Id : \mathcal{M} \to \mathcal{M}$ which simply maps any chord to itself.

Inverses: Similar to closure, one can find inverses for every element in our group. In particular, we have that $P \circ P = L \circ L = R \circ R = Id$, so each function is an inverse of itself.

Hence G is a group as required. □

Conclusion

One can also show the group found in Theorem 1 is known as the *dihedral group* D_n (2), which is the group generated by the symmetries of a regular n-gon (in the case of the $\mathcal{PLR}$ group, we get D_{12}). One can also show this group has a *subgroup* generated by $\mathcal{P}$ and $\mathcal{R}$ which is isomorphic to D_4 (3). These functions give various results in group theory, and can be extended further to consider functions to describe other musical concepts such as transposition of notes.